Examples Computing $\mathcal{L}\{f(t)\}$ using Formulas & Rules

Easy Examples

- $\mathcal{L}\{e^{2t}\sin 3t\}$

$$\mathcal{L}\{e^{2t}\sin 3t\} = \ell^2\left[\mathcal{L}\{\sin 3t\}\right]$$
$$= \ell^2\left[\frac{3}{s^2+9}\right] = \boxed{\frac{3}{(s-2)^2+9}}$$

- $\mathcal{L}\{e^{4t}t^3\}$

$$\mathcal{L}\{e^{4t}t^3\} = \ell^4\left[\mathcal{L}\{t^3\}\right]$$
$$= \ell^4\left[3!/s^4\right] = \boxed{6/(s-4)^4}$$

- $\mathcal{L}\{4+3t^2\}$

$$\mathcal{L}\{4+3t^2\} = 4\mathcal{L}\{1\} + 3\mathcal{L}\{t^2\}$$
$$= \boxed{4\cdot 1/s + 3\cdot 2/s^3}$$

- $\mathcal{L}\{u_2(t)\}$

$$\mathcal{L}\{u_2(t)\} = e^{-2s}\mathcal{L}\{1\}$$
$$= \boxed{e^{-2s}\cdot 1/s}$$

Note: All exponentials in frequency domain have powers with **negative** s. (or no s)

Medium Examples

- $\mathcal{L}\{e^{3t+4}\cos 5t\}$

$$= \mathcal{L}\{e^{3t}\cdot e^{4}\cdot \cos 5t\} = e^4\mathcal{L}\{e^{3t}\cdot\cos 5t\}$$
$$= e^4\ell^3\left[\mathcal{L}\{\cos 5t\}\right] = e^4\ell^3\left[\frac{s}{s^2+25}\right]$$
$$= \boxed{e^4\cdot\frac{(s-3)}{(s-3)^2+25}}$$

- $\mathcal{L}\{e^{t-4}t^2\}$

$$= \mathcal{L}\{e^{t}\cdot e^{-4}\cdot t^2\} = e^{-4}\mathcal{L}\{e^{t}t^2\}$$
$$= e^{-4}\ell^1\left[\mathcal{L}\{t^2\}\right] = e^{-4}\ell^1\left[2/s^3\right]$$
$$= \boxed{e^{-4}\cdot 2/(s-1)^3}$$

- $\mathcal{L}\{2+3u_2(t)\cdot t - u_4(t)\cdot t^2\}$

$$= 2\mathcal{L}\{1\} + 3\mathcal{L}\{u_2(t)\cdot t\} - \mathcal{L}\{u_4(t)\cdot t^2\}$$
$$= 2\cdot 1/s + 3e^{-2s}\mathcal{L}\{(t+2)\} - e^{-4s}\mathcal{L}\{(t+4)^2\}$$
$$= 2/s + 3e^{-2s}\left(\mathcal{L}\{t\} + 2\mathcal{L}\{1\}\right)$$
$$- e^{-4s}\left(\mathcal{L}\{t^2\} + 8\mathcal{L}\{t\} + 16\mathcal{L}\{1\}\right)$$
$$= \boxed{2/s + 3e^{-2s}\left(1/s^2 + 2/s\right) - e^{-4s}\left(2/s^3 + 8/s^2 + 16/s\right)}$$

• $\mathcal{L}\{u_2(t)\cdot e^{5t}\}$

$$\mathcal{L}\{u_2(t)\cdot e^{5t}\} = e^{-2s}\,\mathcal{L}\{e^{5(t+2)}\}$$

$$= e^{-2s}\,\mathcal{L}\{e^{5t}\cdot e^{10}\} = e^{-2s+10}\,\mathcal{L}\{e^{5t}\}$$

$$= \boxed{e^{-2(s-5)}\cdot \frac{1}{(s-5)}}$$

— Alternate Calculation —

$$\mathcal{L}\{u_2(t)\, e^{5t}\} = \ell^{5}\left[\mathcal{L}\{u_2(t)\}\right]$$

$$= \ell^{5}\left[e^{-2s}\cdot 1/s\right]$$

$$= \boxed{e^{-2(s-5)}\cdot \frac{1}{(s-5)}}$$

• $\mathcal{L}\{u_\pi(t)\cdot \sin 2t\}$

$$\mathcal{L}\{u_\pi(t)\cdot \sin 2t\} = e^{-\pi s}\,\mathcal{L}\{\sin 2(t+\pi)\}$$

$$= e^{-\pi s}\,\mathcal{L}\{\sin(2t+2\pi)\}$$

(Note: $\sin(x+2\pi) = \sin x$)

$$= e^{-\pi s}\,\mathcal{L}\{\sin 2t\}$$

$$= \boxed{e^{-\pi s}\cdot \frac{2}{s^2+4}}$$

• $\mathcal{L}\{u_\pi(t)\cdot \cos t\}$

$$\mathcal{L}\{u_\pi(t)\cdot \cos t\} = e^{-\pi s}\,\mathcal{L}\{\cos(t+\pi)\}$$

(Note: $\cos(x+\pi) = -\sin x$)

$$= e^{-\pi s}\,\mathcal{L}\{-\sin t\}$$

$$= \boxed{-e^{-\pi s}\cdot \frac{1}{s^2+1}}$$

Hard Examples

• $\mathcal{L}\{u_4(t)\cdot e^{-t+2}\cdot \sin 3(t-4)\}$

$$= \mathcal{L}\{u_4(t)\cdot e^{-t}\cdot e^{2}\cdot \sin 3(t-4)\}$$

$$= e^{2}\,\mathcal{L}\{u_4(t)\, e^{-t}\cdot \sin 3(t-4)\}$$

$$= e^{2}\,\ell^{-1}\left[\mathcal{L}\{u_4(t)\cdot \sin 3(t-4)\}\right]$$

$$= e^{2}\,\ell^{-1}\left[e^{-4s}\,\mathcal{L}\{\sin 3((t+4)-4)\}\right]$$

$$= e^{2}\cdot e^{-4(s+1)}\,\ell^{-1}\left[\mathcal{L}\{\sin 3t\}\right]$$

$$= e^{-4(s+1)+2}\,\ell^{-1}\left[\frac{3}{s^2+9}\right]$$

$$= \boxed{e^{-4(s+1)+2}\cdot \frac{3}{(s+1)^2+9}}$$

- $\mathcal{L}\{u_2(t)\cdot e^{3t}\cdot \cos 5t\}$

$$\mathcal{L}\{u_2(t)\, e^{3t}\cdot \cos 5t\} = \ell^3\left[\mathcal{L}\{u_2(t)\cdot \cos 5t\}\right]$$
$$= \ell^3\left[e^{-2s}\mathcal{L}\{\cos 5(t+2)\}\right]$$
$$= e^{-2(s-3)}\,\ell^3\left[\mathcal{L}\{\cos(5t+10)\}\right]$$
$$= e^{-2(s-3)}\,\ell^3\left[\mathcal{L}\{(\cos 5t)\cdot(\cos 10) - (\sin 5t)(\sin 10)\}\right]$$
$$= e^{-2(s-3)}\,\ell^3\left[(\cos 10)\mathcal{L}\{\cos 5t\} - (\sin 10)\mathcal{L}\{\sin 5t\}\right]$$
$$= e^{-2(s-3)}\,\ell^3\left[(\cos 10)\cdot\frac{s}{s^2+25} - (\sin 10)\frac{5}{s^2+25}\right]$$
$$= \boxed{e^{-2(s-3)}\left((\cos 10)\frac{s-3}{(s-3)^2+25} - (\sin 10)\frac{5}{(s-3)^2+25}\right)}$$

- $\mathcal{L}\{u_2(t)\cdot e^{3t+1}\cdot t^3\}$

$$\mathcal{L}\{u_2(t)\cdot e^{3t}\cdot e^1\cdot t^3\} = e\,\mathcal{L}\{u_2(t)\, e^{3t}\cdot t^3\}$$
$$= e\,\ell^3\left[\mathcal{L}\{u_2(t)\cdot t^3\}\right]$$
$$= e\,\ell^3\left[e^{-2s}\mathcal{L}\{(t+2)^3\}\right]$$
$$= e\cdot e^{-2(s-3)}\,\ell^3\left[\mathcal{L}\{t^3+6t^2+12t+8\}\right]$$
$$= e^{-2(s-3)+1}\,\ell^3\left[\mathcal{L}\{t^3\}+6\mathcal{L}\{t^2\}+12\mathcal{L}\{t\}+8\mathcal{L}\{1\}\right]$$
$$= e^{-2(s-3)+1}\,\ell^3\left[3!/s^4 + 6\cdot 2/s^3 + 12\cdot 1/s^2 + 8\, 1/s\right]$$
$$= \boxed{e^{-2(s-3)+1}\left(\frac{3!}{(s-3)^4}+\frac{12}{(s-3)^3}+\frac{12}{(s-3)^2}+\frac{8}{(s-3)}\right)}$$

Expert Examples

- $\mathcal{L}\{t\cos 2t\}$

$y = t\cos 2t$ $\qquad y(0)=0$

$y' = \cos 2t - 2t\sin 2t$ $\qquad y'(0)=1$

$y'' = -4\sin 2t - 4t\cos 2t$ $\qquad y''(0)=0$

So $y = t\cos 2t$ is the solution to the differential equation

$$y''+4y = -4\sin 2t \quad \text{with} \begin{cases} y(0)=0 \\ y'(0)=1 \end{cases}$$

$\Downarrow \mathcal{L}$

$$(s^2Y - s\cdot 0 - 1) + 4Y = -4\,\frac{2}{s^2+4}$$
$$(s^2+4)Y = \frac{-8}{s^2+4} + 1$$
$$Y = \frac{1}{s^2+4}\left(\frac{-8+(s^2+4)}{s^2+4}\right)$$
$$= \boxed{\frac{s^2-4}{(s^2+4)^2}}$$

— Alternate Calculation —

$$\mathcal{L}\{t\cos 2t\} = -\frac{d}{ds}\left(\mathcal{L}\{\cos 2t\}\right)$$
$$= -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$
$$= -\frac{(s^2+4) - 2s\cdot s}{(s^2+4)^2} = \boxed{\frac{s^2-4}{(s^2+4)^2}}$$

- $\mathcal{L}\{te^t\cos t\}$

$$y = te^t\cos t \qquad y(0)=0$$

$$y' = e^t\cos t + te^t\cos t - te^t\sin t \qquad y'(0)=1$$

$$\begin{aligned} y'' = {} & e^t\cos t - e^t\sin t \\ & + e^t\cos t + te^t\cos t - te^t\sin t \\ & - e^t\sin t - te^t\sin t - te^t\cos t \\ = {} & 2e^t\cos t - 2e^t\sin t - 2te^t\sin t \end{aligned}$$

So $y = te^t\cos t$ is the solution to the differential equation

$$y'' - 2y' + 2y = -2e^t\sin t \quad \text{with } \begin{cases} y(0)=0 \\ y'(0)=1 \end{cases}$$

$$\Big\downarrow \mathcal{L}$$

$$(s^2Y - s\cdot 0 - 1) - 2(sY - 0) + 2Y = -2\,\frac{1}{(s-1)^2+1}$$

$$((s-1)^2+1) \quad (s^2-2s+2)Y = \frac{-2}{(s-1)^2+1} + 1$$

$$Y = \frac{1}{(s-1)^2+1}\left(\frac{-2+(s-1)^2+1}{(s-1)^2+1}\right)$$

$$= \boxed{\frac{(s-1)^2-1}{((s-1)^2+1)^2}}$$

— Alternate Calculation —

$$\mathcal{L}\{t\,e^t\cos t\} = -\frac{d}{ds}\left(\mathcal{L}\{e^t\cos t\}\right)$$

$$= -\frac{d}{ds}\left(\frac{s-1}{(s-1)^2+1}\right)$$

$$= -\left[\frac{((s-1)^2+1) - 2(s-1)(s-1)}{((s-1)^2+1)^2}\right]$$

$$= -\left[\frac{-(s-1)^2+1}{((s-1)^2+1)^2}\right]$$

$$= \boxed{\frac{(s-1)^2-1}{((s-1)^2+1)^2}}$$

"Yippee ki-yay, motherf**ker!"

— Bruce Willis (Die Hard)